The axiomatic power of Kolmogorov complexity

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Plan

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Kolmogorov complexity

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- Then we define complexity C by $\mathrm{C} = \mathrm{C}_U$
- For prefix free machine the situation is the same. Then we can define prefix free complexity K with an optimal prefix free machines.

Gödel theorem with Kolmogorov complexity

Theorem (Chaitin)

Peano arithmetic is consistent can prove sentences of the form

$$\mathrm{C}(x) \geq n$$

only for a finite number of n.

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- *Then with a program of length $d + \log(n)$ we can find the first string y such that Peano arithmetic proves $\mathrm{C}(y) \geq n$ ...*

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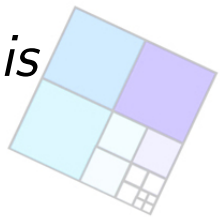
- *Then with a program of length* $d + \log(n)$ *we can find the first string y such that Peano arithmetic proves* $\mathrm{C}(y) \geq n$ *. . .But this short program outputs a string with complexity larger than n*



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- Then we get a stronger theory. . .can we prove the consistency of Peano arithmetic with this new theory ?

A more abstract vision

- More generally we can add more than one axiom :

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  - We can repeat this several times, for each operation we pay δ until the initial capital ε is exhausted

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- This game is not too dangerous . . .

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- And if we add false but non provable axioms ?

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All information about Kolmogorov complexity

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If, in PA, *we add as axioms all true statement of the form*

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Proof

This is mainly an effective version of the reduction of the halting problem to the function C*.*

Most information about Kolmogorov complexity

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Let ψ be an Π_1-unprovable statement.
There is a set A with 99% (for each length) of strings x such that $\mathrm{C}(x) \geq |x|$ and such that theory PA *plus all axiom $\mathrm{C}(x) \geq |x|$ for x in* PA *does not prove ψ*

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Proof

By induction (on the length of strings x) we show that if the previous theorem does not hold then ψ is provable from too many random axioms.

Less is more ?

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There is a sequence of string (x_n) *such that for all* n *we have* $|x_n| = n$ *and* $\mathrm{C}(x_n) \geq n$ *is true and if we add all axioms* $\mathrm{C}(x_n) \geq n$ *to* PA *then we get* PA *plus all true* Π_1*-statements*

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If x_n *is the left most string of length* n *such that* $\mathrm{C}(x_n) \geq n$ *then with this axiom we can find (and prove that this number has this property) a waiting time* t *such that all program of length less than* n *halt in time less than* t*.*

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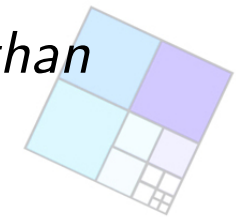
Then with this t *we can decide any* Π_1*-statement of length less than* $n - O(1)$*.*



Just one constant

Theorem

There exists some constant c such that PA *together with all true statements of the type $\mathrm{C}(x|y) \geq c$ proves all true Π_1-statements.*

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- In some sense this phenomenon is accidental because most strings do not have this property
- How can we characterise these strings ? The utility of a string x could be related to $\mathrm{C}^{\emptyset'}(x) - \mathrm{C}(x)$

The same axiom with a partial information

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- Towards a semantic reduction ?

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PA plus a random sequence

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- In the sequels $\mathrm{MLR}_c(X)$ denotes PA plus all axioms $\mathrm{K}(X \upharpoonright n) \geq n - c$

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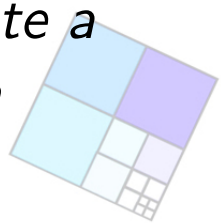
Proof

The proof uses the fact the set of axioms proving ψ (if ψ is not provable in PA*) has small measure and from this fact we can create a Martin-Löf test (with parameter c) showing that X is not random*



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- *The proof use advanced technical tools from recursion theory*
- *We create a incomplete (in the Turing sense) set of strings A such that for all n we have* $X \upharpoonright n \in A$ *ans for all* $x \in A$ *we have* $\mathrm{K}(x) \geq |x| - c$

More than a random sequence

Theorem

For some sequence $X \in 2^\omega$, if we add axioms "$\mathrm{K}(X \upharpoonright n) \geq n - c$ and $X \upharpoonright n$ can be continued in a random string with a deficiency c"
This theory can decide all Π_1-statements

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Proof

If X is a left most path in sequences with deficiency c then if we know that $X \upharpoonright n$ can be continued in a random string with a deficiency c we can prove that X is a left most path in sequences with deficiency c. And this implies that X is a Chaitin's Ω number and from that we can provably decide which programs halt.

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is exactly the set of computable sequences

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- The way to express a property can be very important in the provability world

Program generating an Ω

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- Can we do the same for c.e. K-trivial ?

¿ Questions ?