Effective symbolic dynamics

E. Jeandel

Montpellier, France, World, Universe

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Plan

1. Introduction

2. Introduction

3. Turing degrees
What is effective symbolic dynamics?

- The effective counterpart of symbolic dynamics

Say it again?
Symbolic Dynamics modelize a general dynamical systems by a discrete space.

- Let $f : X \mapsto X$.
- Let $X = \bigcup_{i \in \Sigma} U_i$ a partition of $X$ into clopens.
- To each point $x \in X$, associate its trajectory $\omega(x) \in \Sigma^\mathbb{N}$.

$$\omega(x)_i = j \iff f^i(x) \in U_j$$

- $\{\omega(x), x \in X\}$ is a symbolic dynamical system.

Usually, instead of a partition into clopen sets, we have an “almost” partition, with respect to some invariant measure.
There will be no $\mu$ in this talk.
Definition

A subset \( S \subseteq \Sigma^\mathbb{Z} \) (resp. \( \Sigma^\mathbb{N} \)) is a subshift if it is topologically closed and invariant under the shift.

- The topology is the product topology: \( d(x, y) = 2^{-\min\{|i|, x_i \neq y_i\}} \)
- Shift: \( \sigma(x)_i = x_{i+1} \).

Symbolic Dynamics is (arguably) the study of subshifts.

Alternate definition:

- There exists a language \( L \) of finite words so that \( x \in S \) if and only if \( x \) does not contain any factor in \( L \).
Examples

\[ \Sigma = \{a, b\} \]

- \( L = \emptyset \), \( S = \Sigma^\mathbb{Z} \)
- \( L = \{a, ba, bb\}, S = \emptyset. \)
- \( L = \{ba\}, S = \{\omega a^\omega\} \cup \{\omega b^\omega\} \cup \{\omega ab^\omega\} \)
- \( L = \{xx, x \in \Sigma^+\}, \)
- \( L = \{xyxyx, x, y \in \Sigma^+\}, \)
Examples

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- \( L = \{a, ba, bb\}, S = \emptyset. \)
- \( L = \{ba\}, S = \{\omega a^\omega\} \cup \{\omega b^\omega\} \cup \{\omega ab^\omega\} \)
- \( L = \{xx, x \in \Sigma^+\}, S = \emptyset \)
- \( L = \{xyxyx, x, y \in \Sigma^+\}, S \text{ contains the Thue-Morse sequence} \)
Effective sets (\(\Pi^0_1\) classes of sets)

We are interested in effective symbolic dynamics, which is the theory of effective subshifts.

**Definition**

A set \(S \subseteq \Sigma^\mathbb{N}\) is effectively closed if its complement is the computable union of cylinders.

Alternatively:

- \(S\) is the set of oracles on which a given Turing machine does not halt.
- \(S\) is given by a computable (or c.e.) set of forbidden prefixes.

Effective sets crawl everywhere in recursive mathematics.
Effective symbolic dynamics

Definition

An effective subshift is a subshift which is effectively closed.

All previous examples of subshifts are effective.

- Do effective subshifts exhibit the same complexity as effective sets?
- Do the closure under shift give additional properties?
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Tilings are a more geometric version of effective subshifts.

They are *finitely* presented.

Tilings have more or less the same recursive properties as effective subshifts:

- more on this later.

---
A tileset is given by:

- A finite set of colors $\Sigma$
- A *finite* set of forbidden patterns $P$. 

Forbidden patterns
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Forbidden patterns
Tilings

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Forbidden patterns

![Forbidden patterns diagram]
Tilings

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Tilings

A tileset is given by:

- A finite set of colors $\Sigma$
- A \textit{finite} set of forbidden patterns $P$. 

Forbidden patterns

![Forbidden patterns image]
If $\tau$ is a tileset, let $S_\tau$ be the set of tilings by $\tau$

- $S_\tau$ is a two-dimensional subshift
  - two-dimensional: closed under horizontal and vertical shift
- $S_\tau$ is \textit{of finite type}: it can be given by a \textit{finite} set $L$ of forbidden factors.

In particular $S_\tau$ is effective.

Moreover, $S_\tau$ has recursive properties similar to (one-dimensional) effective subshifts. Why?
Computation inside tilings
Transfer theorem

Theorem (Durand-Romashchenko-Shen, Aubrun-Sablik 2012)

For every effective subshift $S$ over the alphabet $\Sigma$, there exists a tileset $\tau$ over the alphabet $\Sigma \times \Delta$ so that $S$ is exactly the $\Sigma$-component of lines of $S_\tau$.

- Almost all theorems on effective subshfits have a tiling counterpart
### Examples

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By date of publication (Simpson 2012 actually predates Miller 2011, Jeandel-Vanier is contemporary of Cenzer et alii 2011)
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Reminder

- Do effective subshifts exhibit the same complexity as effective sets?
- Do the closure under shift give additional properties?

Here complexity means Turing degrees.
Basis theorems

Basis theorems state that every (nonempty) effective set contains a point with a specific property.

Theorem

- Any effective set contains a point of Turing degree less than or equal to $0'$. (Kreisel 1953)
- Any effective set contains a point of Turing degree less than $0'$. (Shoenfield 1960)
- Any effective set contains a point of hyperimmune-free degree (Jockusch-Soare 1972)
- Any effective set contains a point of low degree (Jockusch-Soare 1972)
- Any effective set . . .
Basis theorems

Basis theorems state that every (nonempty) effective set contains a point with a specific property.

**Theorem**

- Any effective subshift contains a point of Turing degree less than or equal to $0'$. (Kreisel 1953)
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- Any effective subshift contains a point of low degree (Jockusch-Soare 1972)
- Any effective subshift . . .
“There exists an effective set with some specific property”

- Might not be true anymore
- Subshifts have additional properties

What is this additional property?
A subshift is minimal if it contains no proper (nonempty) subshifts.

**Theorem (Birkhoff 1912)**

*Any subshift contains a minimal subshift.*

Essentially Zorn’s/Konig’s lemma + compactness

- The subshift defined by $L = \{ xyxyx, x, y \in \Sigma^+ \}$ is actually minimal.
- Way to obtain a minimal subshift, starting from $L$:
  - For each $w \in \Sigma^+$
    - if the subshift defined by $L \cup \{w\}$ is not empty, $L := L \cup \{w\}$
  - if $S$ is effective, it might contain no minimal effective subshift.
A biinfinite word $w$ is uniformly recurrent if there exists a map $f$ so that any factor of $w$ of length $n$ appears in any window of size $f(n)$ of $w$.

**Theorem**

*Every point of a minimal subshift is uniformly recurrent.*

- Periodic words are uniformly recurrent
- A minimal subshift with no periodic word is of cardinality $2^{\aleph_0}$. 
First consequence

**Theorem (Jeandel-Vanier 2012)**

Let $S$ be a (nonempty) subshift.

- Either $S$ contains a periodic (hence recursive) point
- Or $S$ contains points of any Turing degree $\geq_T a$ for some degree $a$.

(Not a dichotomy)

- Also true if $S$ is not effective
- If $S$ is effective, we can choose $a = 0'$.

This regularity is not true of effective sets (Jockush-Soare 1972).
We can suppose $S$ is minimal

Starting from a uniformly recurrent word $w$ in $S$, and a word $x \in \{0, 1\}^\mathbb{N}$, we will build $f(x)$ so that:

- $f(x)$ is in $S$
- $f(x)$ is computable given $w$ and $x$.
- $x$ is computable given $f(x)$.

If $\deg_T x \geq \deg_T w$, then $\deg_T x = \deg_T f(x)$.

To simplify the exposition, we take $S$ over the alphabet $\{0, 1\}$, and a semiinfinite subshift
Let $u$ be a factor of $w$.

There are more than one way to extend $u$.

   Otherwise $w$ is periodic.

There exists $y$ so that $uy0$ and $uy1$ both appear in $w$.

Gives a way to encode one bit . . .

. . . but no way to decode it without $w$
Look at all appearances of $uy$ in $w$
Some are followed by 0, others are followed by 1.
There must be two consecutive occurrences of $uy$ where the first one is followed by 0, the next by 1.
There must be two consecutive occurrences of $uy$ where the first one is followed by 1, the next by 0.

We can use this to encode a bit, and now we can decode.
Encoding: $f(x) = \lim_{i} u_i$

- $u_{-1} = \epsilon$.
- If $x_{i+1} = 0$, find $u_i y_0 z u_i y_1$ in $w$ and call it $u_{i+1}$
- If $x_{i+1} = 1$, find $u_i y_1 z u_i y_0$ in $w$ and call it $u_{i+1}$

Decoding:

- $u_{-1} = \epsilon$.
- Look at the first two consecutive occurrences of $u_i$ in $f(x)$, and at the first time they differ.
- Call $u_{i+1}$ this word
- If they differ in the order $0, 1$, then $x_i = 0$, otherwise $x_i = 1$
Corollary

There is no way to encode an effective set into an effective subshift preserving the structure of Turing degrees.

We have to choose one of the two evils.

- Allowing arbitrary complex points
- Allowing recursive points
**Definition**

$S \leq_{Mu} S'$ if for every $x \in S'$, there exists $y \in S$ that is computable in $x$.

$S \leq_{Me} S'$ if the transformation from $x$ to $y$ is uniform in $x$.

Informally: If $S =_{Mu} S'$, then the “minimal” Turing degrees of $S$ and $S'$ are the same. In particular $S$ has a recursive point iff $S'$ does.
Every Medvedev degree contains an effective subshift

Theorem (Miller, 2011)

For every effective set $S$ there is an effective subshift $S'$ so that $S \equiv_{Me} S'$.

Informally: There is a way to encode an effective set into an effective subshift, but we will lose something.
Idea of the proof (cheating)

- Suppose we have an effective *minimal* subshift $M$ over an *infinite alphabet* $\mathbb{N}$.
- We can encode $x \in S$ by:

```
6 2 1 2 7 4 6 2 4 4 7 1 4 6 1 6 9 5 5 8 5 7 7 9 3 5 1 7 6 5 4 6 6 3 3 4 4 5 4
```

```
x_6 x_2 x_1 x_2 x_7 x_4 x_6 x_2 x_4 x_7 x_1 x_4 x_6 x_1 x_6 x_9 x_5 x_5 x_8 x_5 x_7 x_7 x_9 x_3 x_5 x_1 x_7 x_6 x_5 x_4 x_6 x_6 x_3 x_3 x_4 x_4 x_5 x_4
```

“Theorem”: Given a minimal subshift $M$ over $\mathbb{N}$, we can encode $S \subseteq \{a, b\}^\mathbb{N}$ into a subshift $S' \subseteq M \times \{a, b\}^\mathbb{N}$ over the alphabet $\mathbb{N} \times \{a, b\}$.
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  x_6 x_2 x_1 x_2 x_7 x_4 x_6 x_2 x_4 x_7 x_1 x_4 x_6 x_1 x_6 x_9 x_5 x_5 x_8 x_5 x_7 x_7 x_7 x_9 x_3 x_5 x_1 x_7 x_6 x_5 x_4 x_6 x_6 x_3 x_3 x_4 x_4 x_5 x_4

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  6 2 1 2 7 4 6 2 4 4 7 1 4 6 1 6 9 5 5 8 5 7 7 9 3 5 1 7 6 5 4 6 6 3 3 4 4 5 4
  $x_6 \, x_2 \, x_1 \, x_2 \, x_7 \, x_4 \, x_6 \, x_2 \, x_4 \, x_7 \, x_1 \, x_4 \, x_6 \, x_1 \, x_6 \, x_9 \, x_5 \, x_5 \, x_8 \, x_5 \, x_7 \, x_7 \, x_9 \, x_3 \, x_5 \, x_1 \, x_7 \, x_6 \, x_5 \, x_4 \, x_6 \, x_6 \, x_3 \, x_3 \, x_4 \, x_4 \, x_5 \, x_4$

  “Theorem”: Given a minimal subshift $M$ over $\mathbb{N}$, we can encode $S \subseteq \{a, b\}^\mathbb{N}$ into a subshift $S' \subseteq M \times \{a, b\}^\mathbb{N}$ over the alphabet $\mathbb{N} \times \{a, b\}$.
Idea of the proof (cheating)

Rules:
- The rules defining $M$.
- $\left\{ i \ldots i \right\}$ is forbidden
- If the prefix $x_1 \ldots x_p$ is forbidden, then $i_{i_1} \ldots i_{i_n}$ is forbidden whenever $[1, p] \subseteq \{ i_1, \ldots, i_n \}$

It is easy to see:
- Given an element of $S$, we can produce an element of $S'$ (if $M$ contains computable elements)
- Given any element of $S'$ we can produce an element of $S$. 
Proof (without cheating)

Thanks to J. Cassaigne

- We start from $M$ that forbids $\{xyxyx, x, y \in \{0, 1\}^+\}$
- It is the subshift “generated by” the Thue-Morse word
- We can produce a infinite set $u_i$ of finite words in $M$ that is prefix-free.
  - $u_n = t^n(00)$ where $t : 0 \mapsto 01, 1 \mapsto 10$
- Encode $x_i$ in the position where $u_i$ appears
  - As $u_i$ is prefix-free, this is well defined

For the Thue-Morse word (which is computable), we know exactly where $u_i$ appears

- $u_i$ appears at position $2^i m$ where $m$ is such that $m$ and $m + 1$ have an even number of 1 in their binary representation.

Note: the positions where no $u_i$ appears are left free.
The other solution is to add recursive points.

**Theorem (Cenzer-Dashti-Toska-Wyman, 2011)**

For every effective set $S$, there exists an effective subshift $S'$ so that $S'$ contains the same Turing degrees as $S$, with the additional degree of recursive points.
Consider the effective subshift over \( \{a, b, 0\} \) that contains:

- \( \omega a^\omega \)
- \( \omega b^\omega \)
- \( \omega a0a^\omega \)
- \( \omega b0a0aa0aaa0aaaa0aaaa\ldots \)

Encode the set \( S \) below the 0 symbols.

\[
\begin{align*}
&b0a0aa0aaa0aaaa0aaaaa0a\ldots \\
&0x_10x_200x_3000x_40000x_500000x_60
\end{align*}
\]

We have added finitely many recursive points (up to shift)
Proof

Consider the effective subshift over \( \{a, b, 0\} \) that contains:

- \( \omega a^\omega \)
- \( \omega b^\omega \)
- \( \omega a0a^\omega \)
- \( \omega b0a0aa0aaa0aaaa0aaaaa\ldots \)

Encode the set \( S \) below the 0 symbols.

\[
\begin{align*}
b & 0 a & 0 a & a & a & a & 0 & a & a & a & a & a & a & a & a & a & a & a & a & a & a & 0 & a & \ldots \\
0x_1 & 0x_2 & 00x_3 & 000x_4 & 0000x_5 & 00000x_6 & 0
\end{align*}
\]

We have added finitely many recursive points (up to shift)
Effective subshifts have Turing properties that are not true in general.

Every encoding of effective sets into effective subshifts cannot preserve Turing degrees, but some structure can be salvaged.
Let $w$ be an infinite word, and $L$ its set of prefixes. $w$ has the same complexity as $L$. 
Let $w$ be an infinite word, and $L$ its language.

$L$ can be more complex than $w$:

- $w_{2^i(2j+1)} = 1$ if the Turing machine $M$ on input $i$ stops after less than $j$ steps.

We can always compute some $w$ given $L$. 
Open problems

- What is the link between the complexity of a word and its language $L$?

For example
- There exists an effective subshift so that any of its minimal subshifts has a language of complexity at least $0'$.
- Every subshift contains a point of degree less than $0'$.

Is it true that every subshift contains a uniformly recurrent point of degree less than $0'$?