Effective symbolic dynamics

E. Jeandel

Montpellier, France, World, Universe

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Plan

1. Introduction
2. Introduction
3. Turing degrees

What is effective symbolic dynamics ?

- The effective counterpart of symbolic dynamics

Say it again ?

Symbolic dynamics (1/2)

Symbolic Dynamics modelize a general dynamical systems by a discrete space.

- Let $f : X \mapsto X$.
- Let $X = \cup_{i \in \Sigma} U_i$ a partition of X into clopens
- To each point $x \in X$, associate its trajectory $\omega(x) \in \Sigma^{\mathbb{N}}$

$$\omega(x)_i = j \leftrightarrow f^i(x) \in U_j$$

- $\{\omega(x), x \in X\}$ is a symbolic dynamical system

Usually, instead of a partition into clopen sets, we have an “almost” partition, with respect to some invariant measure
There will be no μ in this talk.

Symbolic Dynamics (2/2)

Definition

A subset $S \subseteq \Sigma^{\mathbb{Z}}$ (resp. $\Sigma^{\mathbb{N}}$) is a subshift if it is topologically closed and invariant under the shift.

- The topology is the product topology: $d(x, y) = 2^{-\min\{|i|, x_i \neq y_i\}}$
- Shift: $\sigma(x)_i = x_{i+1}$.

- Symbolic Dynamics is (arguably) the study of subshifts.

Alternate definition:

- There exists a language L of finite words so that $x \in S$ if and only if x does not contain any factor in L.

Examples

$\Sigma = \{a, b\}$

- $L = \emptyset$, $S = \Sigma^{\mathbb{Z}}$
- $L = \{a, ba, bb\}$, $S = \emptyset$.
- $L = \{ba\}$, $S = \{{}^{\omega}a^{\omega}\} \cup \{{}^{\omega}b^{\omega}\} \cup \{{}^{\omega}ab^{\omega}\}$
- $L = \{xx, x \in \Sigma^{+}\}$,
- $L = \{xyxyx, x, y \in \Sigma^{+}\}$,

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- $L = \{ba\}$, $S = \{^\omega a^\omega\} \cup \{^\omega b^\omega\} \cup \{^\omega ab^\omega\}$
- $L = \{xx, x \in \Sigma^+\}$, $S = \emptyset$
- $L = \{xyxyx, x, y \in \Sigma^+\}$, S contains the Thue-Morse sequence

Effective sets (Π^0_1 classes of sets)

We are interested in effective symbolic dynamics, which is the theory of effective subshifts.

Definition

A set $S \subseteq \Sigma^{\mathbb{N}}$ is effectively closed if its complement is the computable union of cylinders.

Alternatively:

- S is the set of oracles on which a given Turing machine does not halt.
- S is given by a computable (or c.e.) set of forbidden prefixes.

Effective sets crawl everywhere in recursive mathematics.

Effective symbolic dynamics

Definition

An effective subshift is a subshift which is effectively closed

All previous examples of subshifts are effective.

- Do effective subshifts exhibit the same complexity as effective sets ?
- Do the closure under shift give additional properties ?

Plan

Tilings

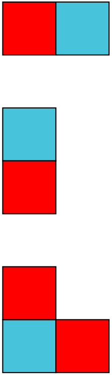
- Tilings are a more geometric version of effective subshifts
- They are *finitely* presented
- Tilings have more or less the same recursive properties as effective subshifts
  - more on this later

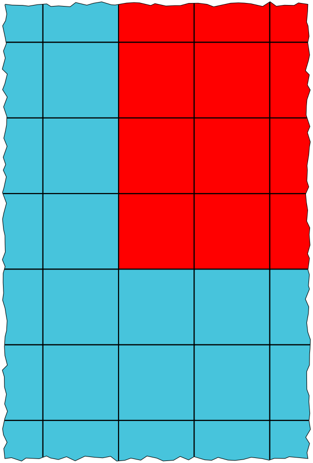
Tilings

A tileset is given by:

- A finite set of colors Σ
- A *finite* set of forbidden patterns P.

Forbidden patterns





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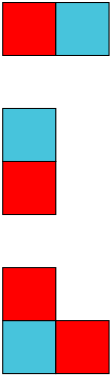

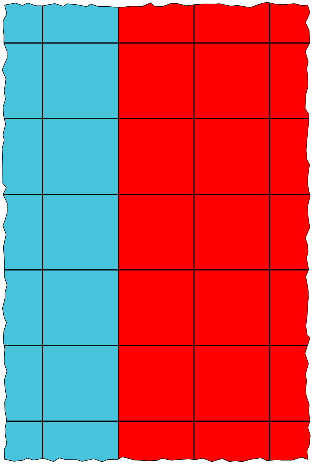

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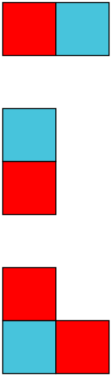

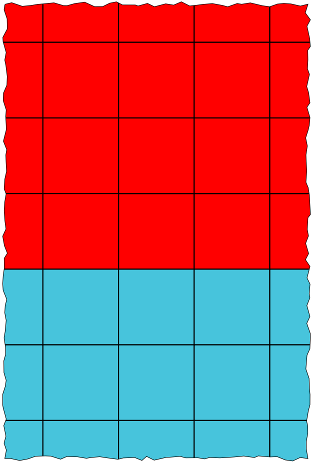

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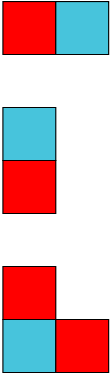

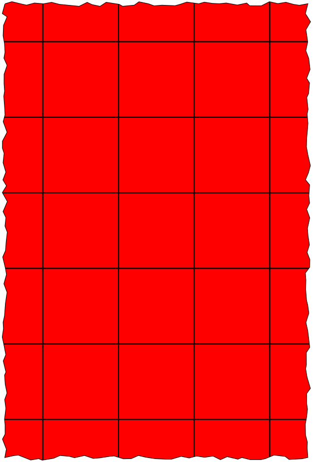

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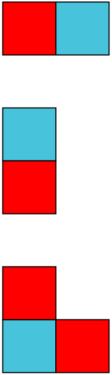
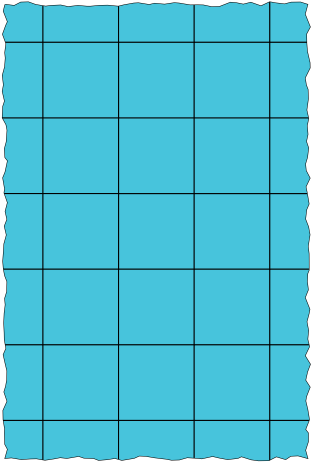




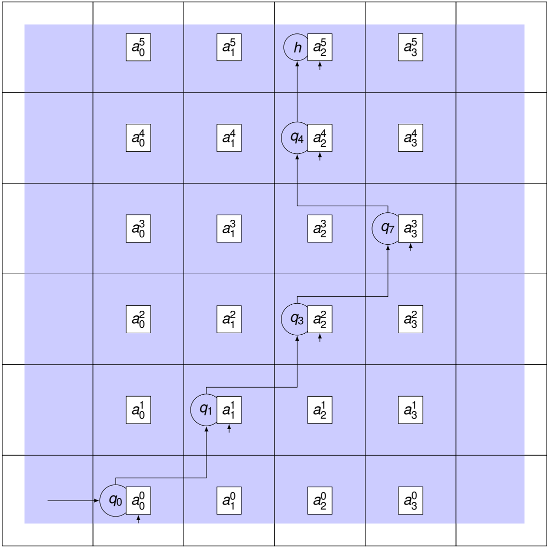
Tilings as subshifts

If τ is a tileset, let S_τ be the set of tilings by τ

- S_τ is a two-dimensional subshift
  - two-dimensional: closed under horizontal and vertical shift
- S_τ is *of finite type*: it can be given by a *finite* set L of forbidden factors.
- In particular S_τ is effective.

Moreover, S_τ has recursive properties similar to (one-dimensional) effective subshifts. Why ?

Computation inside tilings



Transfer theorem

Theorem (Durand-Romashchenko-Shen, Aubrun-Sablik 2012)

For every effective subshift S over the alphabet Σ,
there exists a tileset τ over the alphabet $\Sigma \times \Delta$ so that
S is exactly the Σ-component of lines of S_τ.

- Almost all theorems on effective subshifts have a tiling counterpart

Examples

	ESS	Tilings
subshift with no computable points	Cenzer-Dashti-King, 2008	Myers 1974
entropy can be any right computable real	Hertling-Spandl, 2008	Hochman-Meyerovitch 2010
countable subshifts	Cenzer et alii 2010	Ballier-Durand-Jeandel 2008
This talk	Jeandel-Vanier 2012	Jeandel-Vanier 2012
	Miller 2011	Simpson 2012
	Cenzer et alii 2011	Jeandel-Vanier 2012

By date of publication (Simpson 2012 actually predates Miller 2011, Jeandel-Vanier is contemporary of Cenzer et alii 2011)

Plan

Reminder

- Do effective subshifts exhibit the same complexity as effective sets ?
- Do the closure under shift give additional properties ?

Here complexity means Turing degrees.

Basis theorems

Basis theorems state that every (nonempty) effective set contains a point with a specific property

Theorem

- *Any effective set contains a point of Turing degree less than or equal to* $0'$*. (Kreisel 1953)*
- *Any effective set contains a point of Turing degree less than* $0'$*. (Shoenfield 1960)*
- *Any effective set contains a point of hyperimmune-free degree (Jockusch-Soare 1972)*
- *Any effective set contains a point of low degree (Jockusch-Soare 1972)*
- *Any effective set . . .*

Basis theorems

Basis theorems state that every (nonempty) effective set contains a point with a specific property

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- *Any effective subshift . . .*

Antibasis theorem

"There exists an effective set with some specific property"

- Might not be true anymore
- Subshifts have additional properties

What is this additional property ?

Additional property

- A subshift is minimal if it contains no proper (nonempty) subshift

Theorem (Birkhoff 1912)

Any subshift contains a minimal subshift.

Essentially Zorn's/Konig's lemma + compactness

- The subshift defined by $L = \{xyxyx, x, y \in \Sigma^+\}$ is actually minimal
- Way to obtain a minimal subshift, starting from L:
  - For each $w \in \Sigma^+$
    - if the subshift defined by $L \cup \{w\}$ is not empty, $L := L \cup \{w\}$
- if S is effective, it might contain no minimal effective subshift

Additional property (2)

Definition

A biinfinite word w is uniformly recurrent if there exists a map f so that any factor of w of length n appears in any window of size $f(n)$ of w.

Theorem

Every point of a minimal subshift is uniformly recurrent.

- Periodic words are uniformly recurrent
- A minimal subshift with no periodic word is of cardinality $2^{\aleph_0}$.

First consequence

Theorem (Jeandel-Vanier 2012)

Let S be a (nonempty) subshift.

- *Either S contains a periodic (hence recursive) point*
- *Or S contains points of any Turing degree $\geq_T$ **a** for some degree **a**.*

(Not a dichotomy)

- Also true if *S* is not effective
- If *S* if effective, we can choose $\mathbf{a} = 0'$.

This regularity is not true of effective sets (Jockush-Soare 1972).

Idea of the proof (1/3)

- We can suppose S is minimal
- Starting from a uniformly recurrent word w in S, and a word $x \in \{0,1\}^{\mathbb{N}}$, we will build $f(x)$ so that:
  - $f(x)$ is in S
  - $f(x)$ is computable given w and x.
  - x is computable given $f(x)$.
- If $\deg_T x \geq \deg_T w$, then $\deg_T x = \deg_T f(x)$.

To simplify the exposition, we take S over the alphabet $\{0,1\}$, and a semiinfinite subshift

Idea of the proof (2/3)

- Let u be a factor of w.
- There are more than one way to extend u.
  - Otherwise w is periodic.
- There exists y so that $uy0$ and $uy1$ both appear in w.
- Gives a way to encode one bit…
- … but no way to decode it without w

Idea of the proof (3/3)

- Look at all appearances of uy in w
- Some are followed by 0, others are followed by 1.
- There must be two consecutive occurences of uy where the first one is followed by 0, the next by 1.
- There must be two consecutive occurences of uy where the first one is followed by 1, the next by 0.

We can use this to encode a bit, and now we can decode.

Proof

Encoding: $f(x) = \lim u_i$

- $u_{-1} = \epsilon$.
- if $x_{i+1} = 0$, find [666] a factor $u_i y 0 z u_i y 1$ in w and call it u_{i+1}
- if $x_{i+1} = 1$, find [666] a factor $u_i y 1 z u_i y 0$ in w and call it u_{i+1}

Decoding:

- $u_{-1} = \epsilon$.
- Look at the first two consecutive occurences of u_i in $f(x)$, and at the first time they differ.
- Call u_{i+1} this word
- If they differ in the order $0, 1$, then $x_i = 0$, otherwise $x_i = 1$

[666] formally, the two occurences of u_i may overlap

Corollary

Corollary

There is no way to encode an effective set into an effective subshift preserving the structure of Turing degrees

We have to choose one of the two evils.

- Allowing arbitrary complex points
- Allowing recursive points

Medvedev and Muchnik degrees

Definition

$S \leq_{Mu} S'$ if for every $x \in S'$, there exists $y \in S$ that is computable in x.
$S \leq_{Me} S'$ if the transformation from x to y is uniform in x.

Informally: If $S =_{Mu} S'$, then the "minimal" Turing degrees of S and S' are the same. In particular S has a recursive point iff S' does.

Every Medvedev degree contains an effective subshift

Theorem (Miller, 2011)

For every effective set S there is an effective subshift S' so that $S =_{Me} S'$.

Informally: There is a way to encode an effective set into an effective subshift, but we will lose something.

Idea of the proof (cheating)

- Suppose we have an effective *minimal* subshift M over an *infinite alphabet* $\mathbb{N}$.
- We can encode $x \in S$ by:

6 2 1 2 7 4 6 2 4 4 7 1 4 6 1 6 9 5 5 8 5 7 7 7 9 3 5 1 7 6 5 4 6 6 3 3 4 4 5 4

$x_6 x_2 x_1 x_2 x_7 x_4 x_6 x_2 x_4 x_4 x_7 x_1 x_4 x_6 x_1 x_6 x_9 x_5 x_5 x_8 x_5 x_7 x_7 x_7 x_9 x_3 x_5 x_1 x_7 x_6 x_5 x_4 x_6 x_6 x_3 x_3 x_4 x_4 x_5 x_4$

"Theorem": Given a minimal subshift M over $\mathbb{N}$, we can encode $S \subseteq \{a, b\}^{\mathbb{N}}$ into a subshift $S' \subseteq M \times \{a, b\}^{\mathbb{N}}$ over the alphabet $\mathbb{N} \times \{a, b\}$

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Idea of the proof (cheating)

Rules:

- The rules defining M.
- $\left\{ \begin{array}{ccc} i & \dots & i \\ x & \dots & y \end{array}, x \neq y \right\}$ is forbidden
- If the prefix $x_1 \dots x_p$ is forbidden, then $\begin{array}{ccc} i_1 & \dots & i_n \\ x_{i_1} & \dots & x_{i_n} \end{array}$ is forbidden whenever $[1, p] \subseteq \{i_1, \dots, i_n\}$

It is easy to see:

- Given an element of S, we can produce an element of S' (if M contains computable elements)
- Given any element of S' we can produce an element of S.

Proof (without cheating)

Thanks to J. Cassaigne

- We start from M that forbids $\{xyxyx, x, y \in \{0,1\}^+\}$
- It is the subshift "generated by" the Thue-Morse word
- We can produce a infinite set u_i of finite words in M that is prefix-free.
  - $u_n = t^n(00)$ where $t : 0 \mapsto 01, 1 \mapsto 10$
- Encode x_i in the position where u_i appears
  - As u_i is prefix-free, this is well defined

For the Thue-Morse word (which is computable), we know exactly where u_i appears

- u_i appears at position $2^i m$ where m is such that m and $m+1$ have an even number of 1 in their binary representation.

Note: the positions where no u_i appears are left free.

Second evil

- The other solution is to add recursive points.

Theorem (Cenzer-Dashti-Toska-Wyman, 2011)

For every effective set S, there exists an effective subshift S' so that S' contains the same Turing degrees as S, with the additional degree of recursive points.

Proof

Consider the effective subshift over $\{a, b, 0\}$ that contains:

- ${}^{\omega}a^{\omega}$
- ${}^{\omega}b^{\omega}$
- ${}^{\omega}a0a^{\omega}$
- ${}^{\omega}b0a0aa0aaa0aaaa0aaaa\ldots$

Encode the set S below the 0 symbols.

$$\begin{array}{l} b\,0\,a\,0\,aa\,0\,aaa\,0\,aaaa\,0\,aaaaa\,0\,a\ldots \\ 0x_1 0x_2 00x_3 000x_4 0000x_5 00000x_6 0 \end{array}$$

We have added finitely many recursive points (up to shift)

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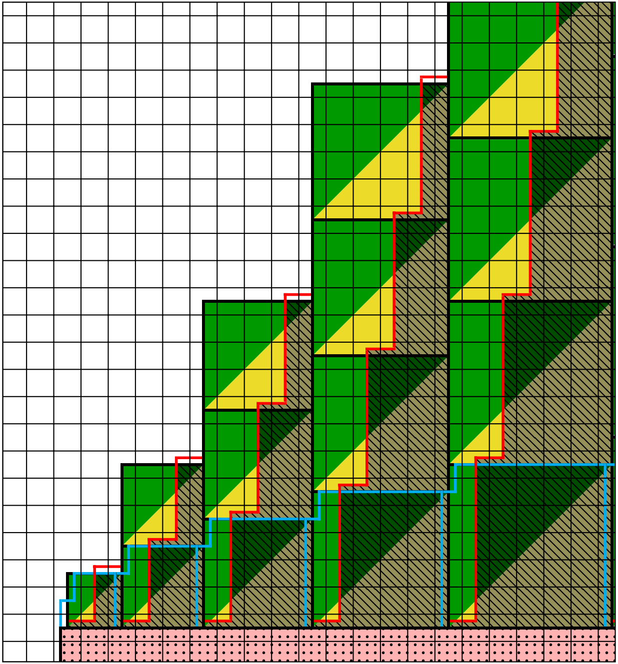

Partial conclusion

- Effective subshifts have Turing properties that are not true in general
- Every encoding of effective sets into effective subshifts cannot preserve Turing degrees, but some structure can be salvaged

Open problem

- Let w be an infinite word, and L its set of prefixes
- w has the same complexity as L

Open problem

- Let w be an infinite word, and L its language
- L can be more complex than w
  - $w_{2^i(2j+1)} = 1$ if the Turing machine M on input i stops after less than j steps.
- We can always compute some w given L

Open problems

- What is the link between the complexity of a word and its language L ?

For example

- There exists an effective subshift so that any of its minimal subshifts has a language of complexity at least $0'$.
- Every subshift contains a point of degree less than $0'$.

Is it true that every subshift contains a uniformly recurrent point of degree less than $0'$?