Computability and ergodic theory

Mathieu Hoyrup

Loria — Laboratoire lorrain de recherche en informatique et ses applications

Inria — informatiques mathématiques

Ergodic decomposition

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  $P[w] = P[0w] + P[1w]$.

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  exists.
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Question

Reading more and more bits of x, can one compute μ_x from x?

Ergodic decomposition

There are two cases:

- μ_x is the same for almost all x's. In that case, $\mu_x = P$ almost surely. P is said to be ***ergodic***.
- μ_x depends on x.

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P is not ergodic $\iff$ it can be decomposed into

$$P = \lambda P_1 + (1 - \lambda) P_2$$

with $P_1 \neq P_2$ shift-invariant and $0 < \lambda < 1$.

Ergodic decomposition

Observation

In general μ_x cannot be uniformly computed from x.

Example

Let $P = \frac{1}{2}(B_p + B_q)$ with $0 < p \neq q < 1$.
Every finite sequence is compatible with B_p and B_q so one can never determine whether $\mu_x = B_p$ or $\mu_x = B_q$.

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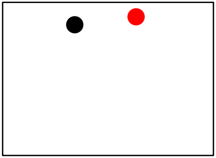
Definition

Let P be a computable shift-invariant measure. P is ***effectively decomposable*** if there is a machine M such that for every $\epsilon > 0$,

$$P\{x : M^x(\epsilon) \text{ computes } \mu_x\} \geq 1 - \epsilon.$$

Ergodic decomposition

The Pólya urn



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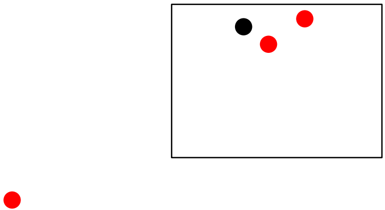

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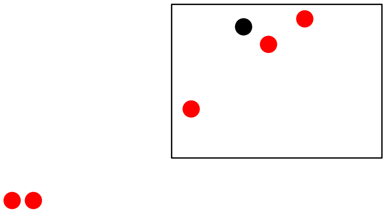

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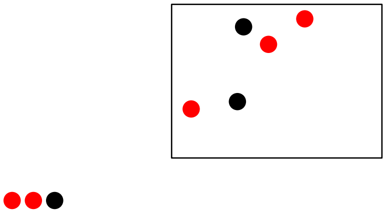

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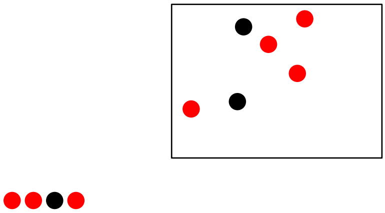

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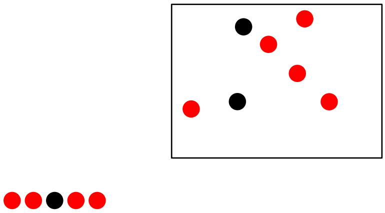

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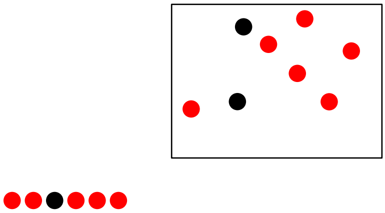

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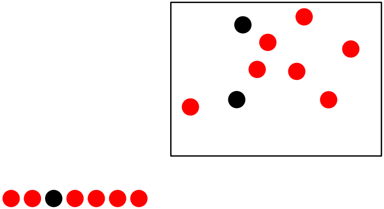

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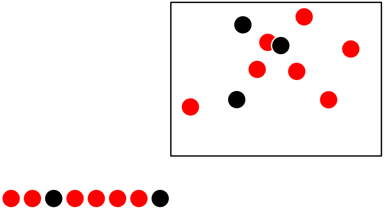

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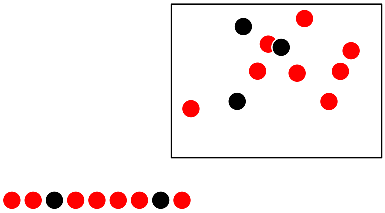

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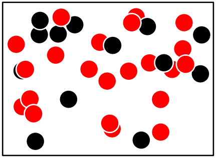

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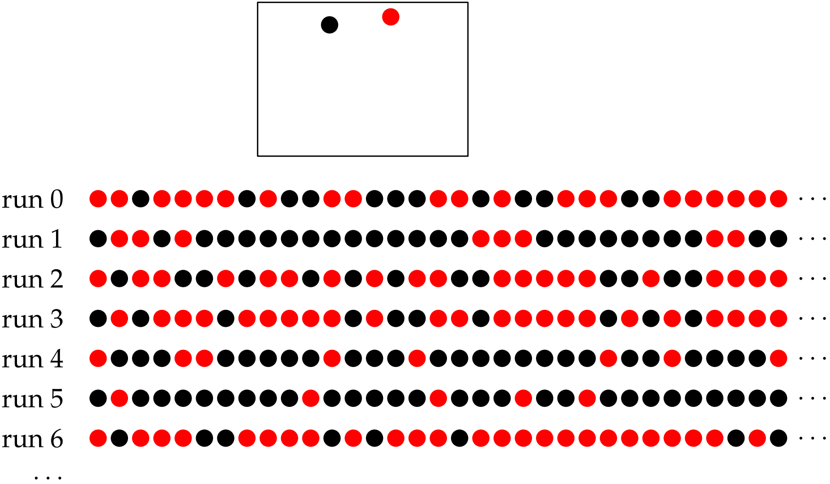

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$$P({\color{red}\bullet}{\color{red}\bullet}{\color{black}\bullet}{\color{red}\bullet}{\color{black}\bullet}{\color{red}\bullet}) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{3}{5}\cdot\frac{2}{6}\cdot\frac{4}{7} = \frac{4!\cdot 2!}{7!}$$

[1] convention: $0! = 1$

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and more generally for $w \in \{\color{red}\bullet\color{black}, \bullet\}^*$,

$$P(w) = \frac{R!\cdot B!}{(R+B+1)!} = \frac{R!\cdot B!}{(|w|+1)!}$$

where R is the number of $\color{red}\bullet$'s and B the number of $\bullet$'s in w. [1]

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Question

What does μ_x look like? Can it be computed from x?

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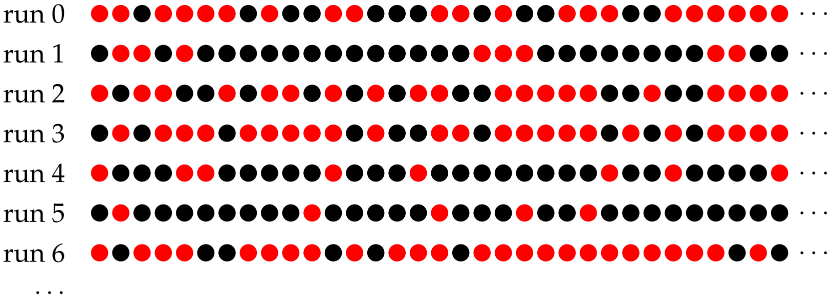

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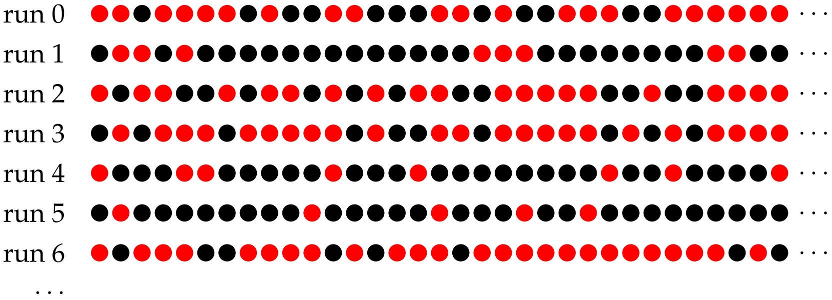

Each run is equivalent to tossing a coin with some particular bias p, chosen uniformly at random in $[0, 1]$.

$$P(w) = \frac{R! \cdot B!}{(R + B + 1)!} = \int_0^1 p^R (1 - p)^B \, \mathrm{d}p$$

Effective topology on measures

- The space of probability measures over Ω with the metric

$$d(P,Q) = \sum_{w \in \{0,1\}^*} 2^{-|w|} |P[w] - Q[w]|$$

is a compact metric space, hence a **Baire space**.

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- The subset of shift-invariant measures is closed:

$$P \in \mathscr{S} \iff \forall w, P[0w] + P[1w] = P[w]$$

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  so it is a compact metric subspace, hence a **Baire space**.
- In $\mathscr{S}$, the set $\mathscr{E}$ of ergodic measures is a **dense Π^0_2-set**:
  - $P \notin \mathscr{E} \iff \exists P_0, P_1 \in \mathscr{S}$ such that $P_0 \neq P_1$ and $P = \dfrac{P_0 + P_1}{2}$
  - The Markovian ergodic measures are dense in $\mathscr{S}$

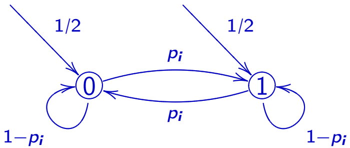
$\mathscr{E}$ is co-meager in $\mathscr{S}$

Ergodic decomposition

Theorem (V'yugin, 1997)

There exists a computable shift-invariant measure P which is not effectively decomposable.

- First step: pick $i \in \{1, 2, 3, \ldots\}$ with probability 2^{-i}.
  Let $p_i = 2^{-t_i}$ where t_i is the halting time of Turing machine M_i ($p_i = 0$ when M_i does not halt).
- Following steps: run the following Markov chain



Let $\epsilon < \hbar/2$. Run the test $M^{\overline{0}}(\epsilon)[0] > 3/4$. It eventually halts and has read a finite number n of bits of $\overline{0}$.

$$P\{\overline{0}\} < P[0^n] \leq P\{\overline{0}\} + \epsilon$$

so $P[0^n]$ is an ϵ-approximation of $\hbar/2$.

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- What about the finite case?

Let $P \neq Q$ be ergodic. Assume $R = \frac{P+Q}{2}$ is computable.

Proposition

R is effectively decomposable $\iff$ P and Q are computable.

Proof.

- If P and Q are computable, the speed of convergence can be computed for P and Q.
- If R is effectively decomposable then using M one can compute P and Q.

□

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Remark

Without the assumption that P, Q are ergodic, it is (too) easy.
Take a non-computable $\lambda \in (0,1)$ and let

$$P = \lambda\delta_{\overline{0}} + (1-\lambda)\delta_{\overline{1}},$$
$$Q = (1-\lambda)\delta_{\overline{0}} + \lambda\delta_{\overline{1}}.$$

$\frac{1}{2}(P+Q) = \frac{1}{2}(\delta_{\overline{0}} + \delta_{\overline{1}})$ is computable, contrary to P and Q.

Theorem (H., 2011)

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The reason is ***topological***: the splitting map $\frac{P+Q}{2} \mapsto \{P, Q\}$ is not continuous.

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The reason is ***topological***: the splitting map $\frac{P+Q}{2} \mapsto \{P, Q\}$ is not continuous.

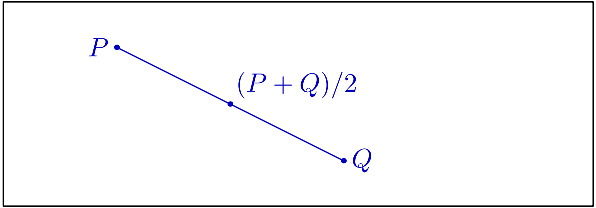
The splitting map is even discontinuous at ***every*** $\frac{P+Q}{2}$ with $P \neq Q$ (and P, Q ergodic).

... and can be proved to be continuous at every ergodic measure $P = \frac{P+P}{2}$.

Proposition

The splitting map is discontinuous at ***every*** $\frac{P+Q}{2}$ *with* $P \neq Q$ *(and* P, Q *ergodic).*

Proof.



Start with $P \neq Q$ are ergodic.

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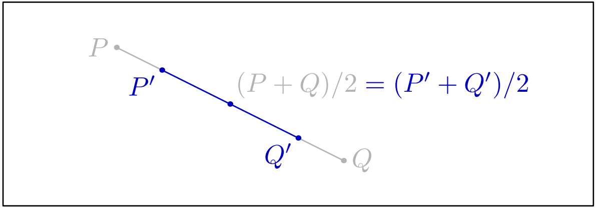

Let $0 < \lambda < 1$ and

$$P' := \lambda P + (1-\lambda) Q,$$
$$Q' := (1-\lambda) P + \lambda Q.$$

$\frac{P'+Q'}{2} = \frac{P+Q}{2}$ but P' and Q' are not ergodic!

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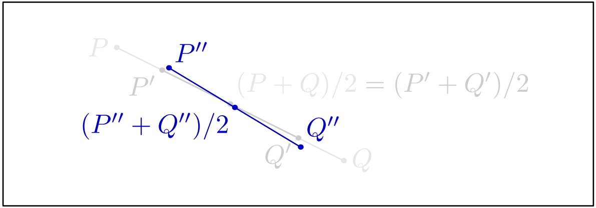
Proof.



Take P'', Q'' ergodic such that

$$P'' \approx P',$$
$$Q'' \approx Q'.$$

Hence $\frac{P''+Q''}{2} \approx \frac{P+Q}{2}$. □

Theorem

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Let $\mathscr{S}$ be the subspace of shift-invariant measures. We actually prove that the set

$$T := \Big\{(P,Q) \in \mathscr{S}^2 : P \text{ and } Q \text{ are ergodic} \\ \text{and not computable relative to } \frac{P+Q}{2}\Big\}$$

is co-meager in $\mathscr{S}^2$, i.e. contains a dense G_δ-set.

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As $\mathscr{S} \times \mathscr{S}$ is a Baire space, it implies that T is non-empty.

Lemma

Let M be a Turing machine. The set

$$C_M := \{(P, Q) \in \mathscr{S}^2 : M^{\frac{P+Q}{2}} \text{ computes } P\}$$

is nowhere dense in $\mathscr{S}^2$.

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Proof.

Let U, V be two open sets of measures. We prove that C_M is not dense in $U \times V$.

- If $U \times V$ is disjoint from C_M, we are done;
- otherwise let $(P, Q) \in C_M \cap (U \times V)$.

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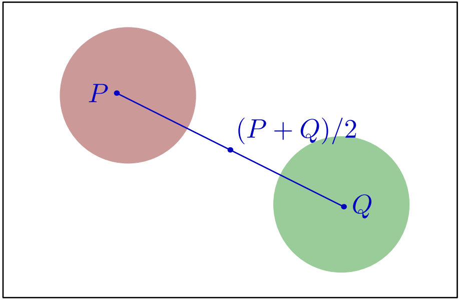

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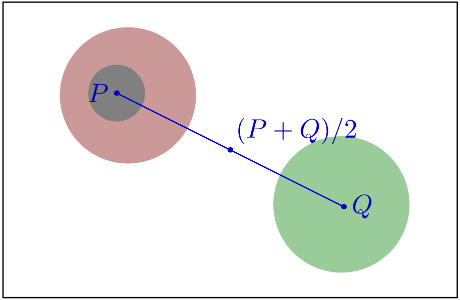

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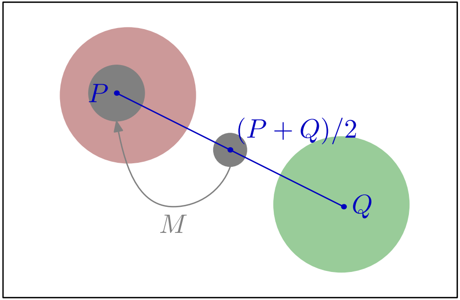

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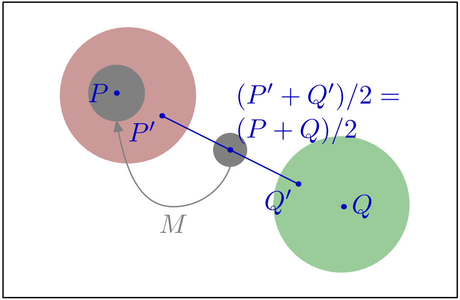
Proof (cont'd).

- Let $(P, Q) \in C_M \cap (U \times V)$.
- $M^{\frac{P'+Q'}{2}}$ does not compute P'



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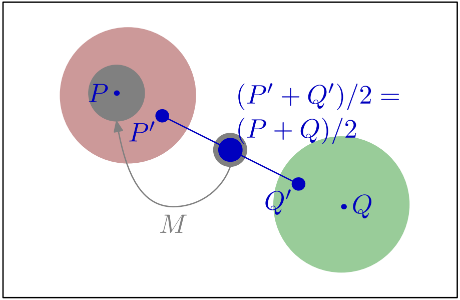
Proof (cont'd).

- Let $(P, Q) \in C_M \cap (U \times V)$.
- $M^{\frac{P'+Q'}{2}}$ does not compute P'
- $M^{\frac{P''+Q''}{2}}$ does not compute P'' for all $P'' \approx P'$ and $Q'' \approx Q'$.

□



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As a result, the set

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is meager.

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$$\{(P, Q) \in \mathscr{S}^2 : P \text{ and } Q \text{ are not computable relative to } P + Q\}$$

is co-meager.

Reminder

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To conclude, the set

$$\{(P, Q) : P \text{ and } Q \text{ are ergodic and not computable relative to } P + Q\}$$

is co-meager.

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There exist P, Q ergodic and non-computable such that $P + Q$ is computable.

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- $P + Q$ is computable,
- P is not computable,
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In this proof, we will say that the machine M computes P if for every ball B of measures,

$$M(B) \downarrow \iff P \in B.$$

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- $B_n \subseteq U_n$,
- the radius of B_n tends to 0.

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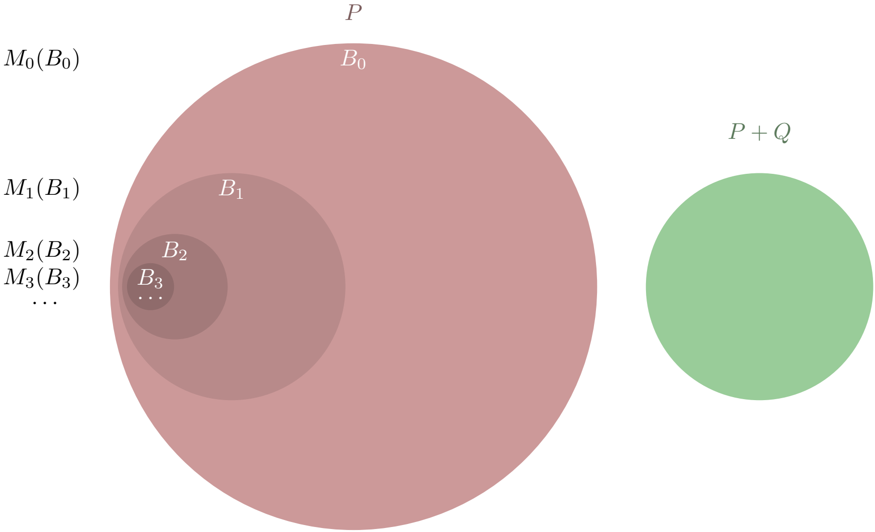
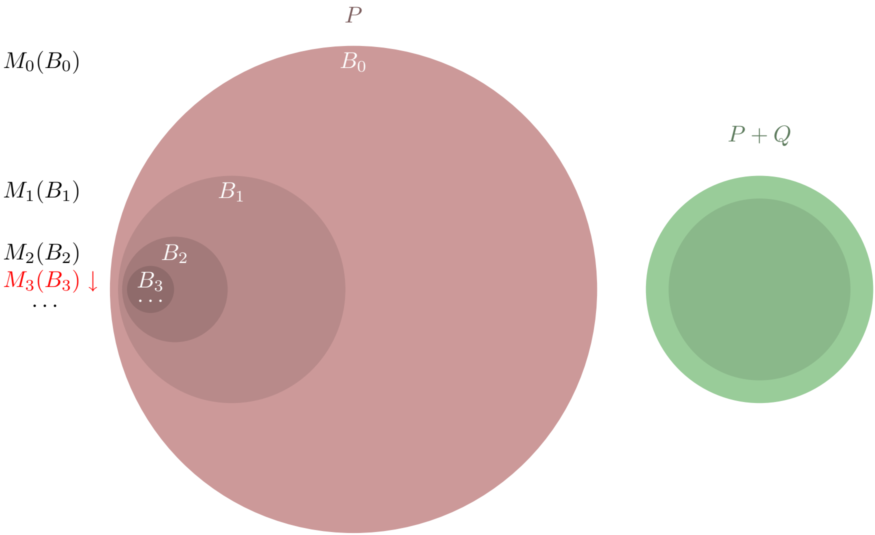
We define a sequence of balls B_n such that

- $\overline{B}_{n+1} \subseteq B_n$,
- $B_n \subseteq U_n$,
- the radius of B_n tends to 0.

P will be the unique member of $\bigcap_n B_n$. It will be automatically ergodic.



At each stage, $\overline{B}_{n+1} \subseteq B_n$ and $B_n \subseteq U_n$ for all n.



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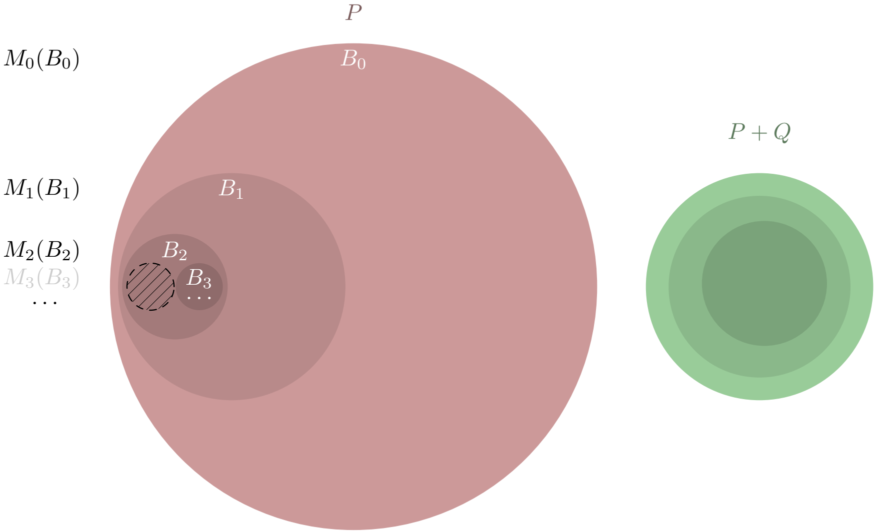

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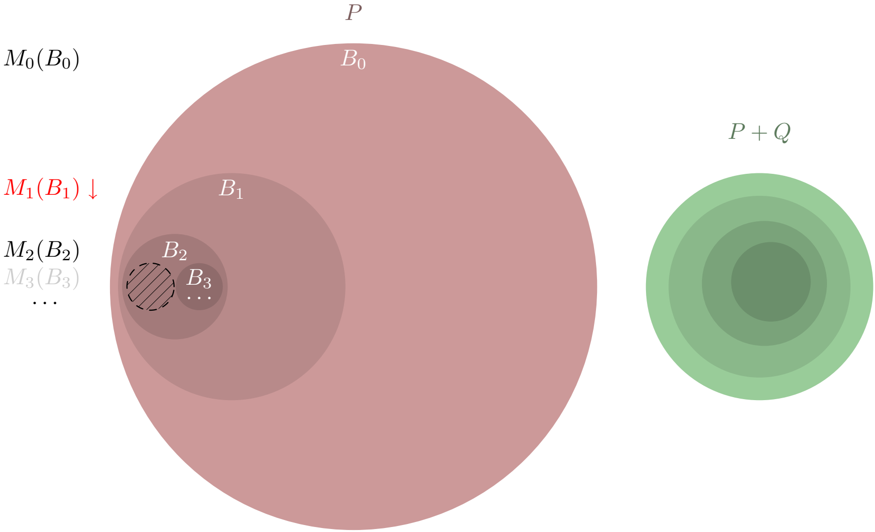

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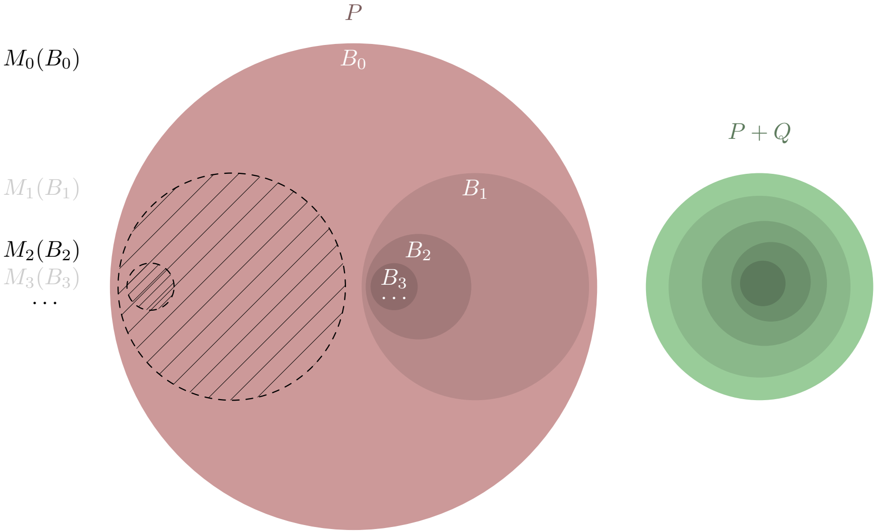

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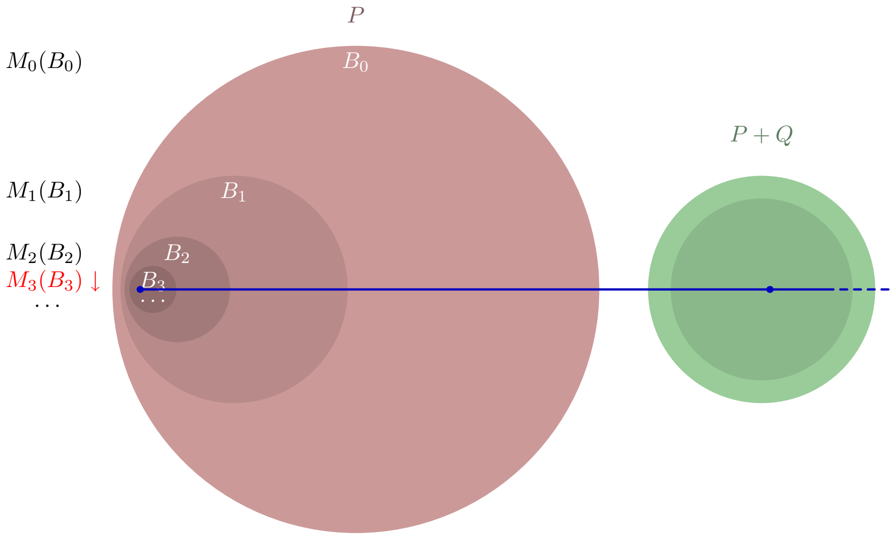

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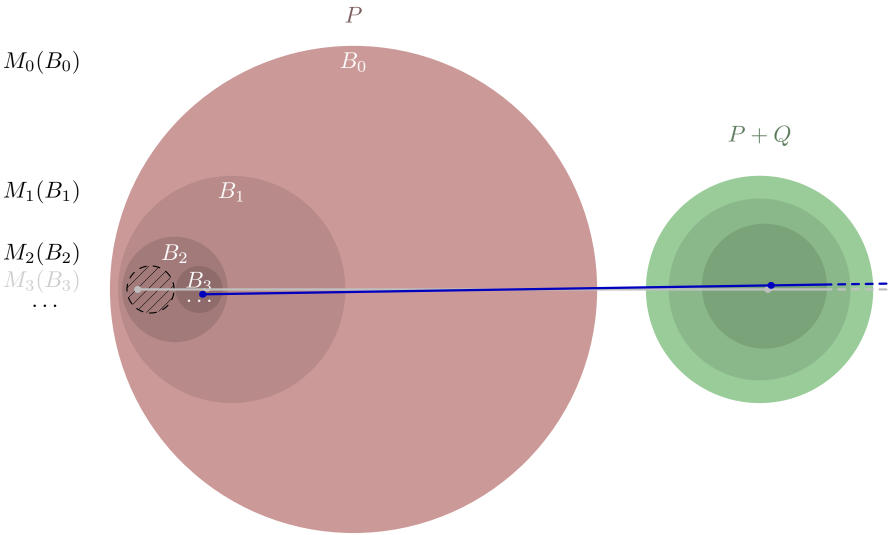

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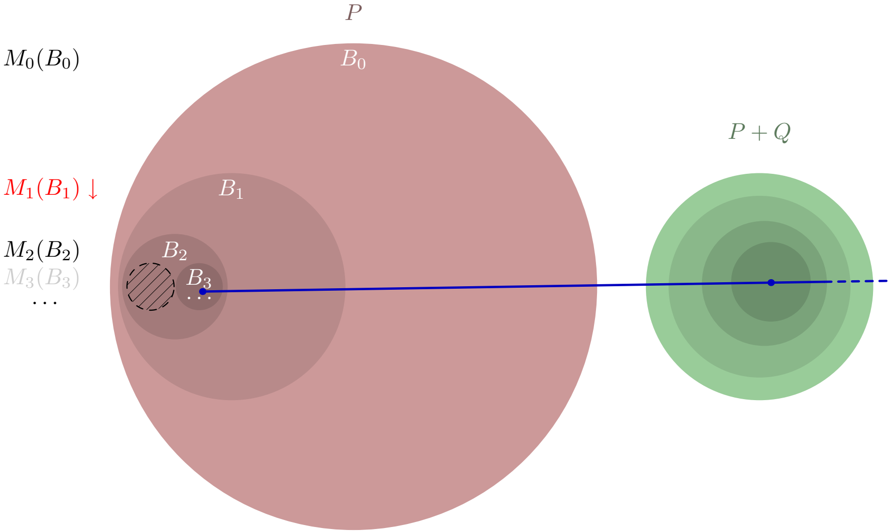

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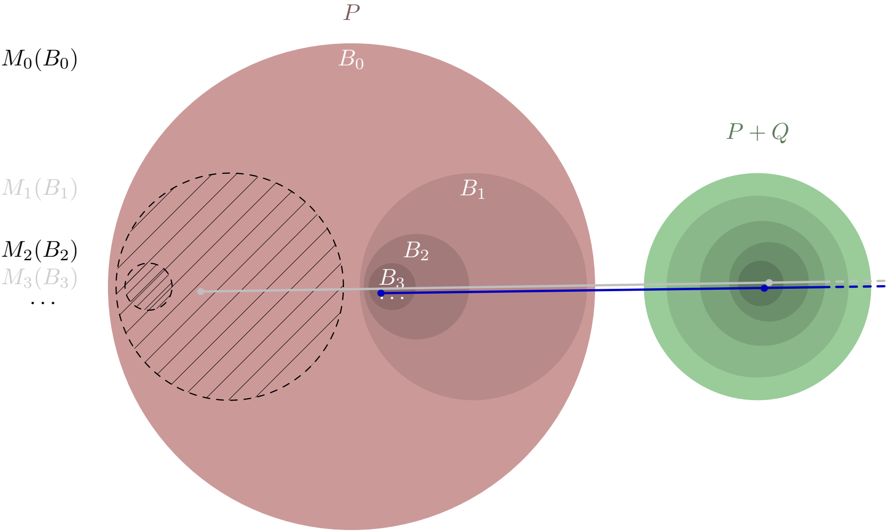

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