Logic, Complexity, and Infinite Computations

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Complexity of finite computations

Complexity of finite computations is often measured by the amount of time or space needed to accept a word of length n.

P = DTIME(Pol)

NP = NTIME(Pol)

P = NP ?

Languages of finite words accepted by different finite machines

- A regular language (accepted by a finite automaton) is in the class **DTIME(n)**.
- A 1-counter language or a context-free language is in the class **DTIME (n^3)**.
- There are recursive languages, accepted by Turing machines, in the class **DTIME(2^n)$\setminus$ P**.
- There are recursive languages, accepted by Turing machines, which are non elementary. For instance Büchi's procedure (1962) to decide whether a monadic second order formula of size n of $S1S$ is true in the structure $(\omega, <)$ might run in time $\underbrace{2^{2^{\cdot^{\cdot^{2^n}}}}}_{\mathbf{O(n)}}$, Moreover Meyer (1975) proved that one cannot essentially improve this result: the monadic second order theory of $(\omega, <)$ is not elementary recursive.

Acceptance of infinite words

- **In the sixties**,
  Acceptance of infinite words by finite automata was firstly considered by **Büchi** in order to study the decidability of the monadic second order theory S1S of one successor over the integers.
- Since then ω-regular languages accepted by Büchi automata and their extensions have been much studied and used for **specification and verification of non terminating systems**.

Büchi acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of final states $F \subseteq K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff there is some state $q_f \in F$ appearing infinitely often during the reading of σ.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

An ω-language $L \subseteq \Sigma^\omega$ is accepted by $\mathcal{A}$ if it is the set of infinite words $\sigma \in \Sigma^\omega$ accepted by $\mathcal{A}$.

Muller acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of accepting sets of states $\mathcal{F} \subseteq 2^K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff the set of states appearing infinitely often during this run is an accepting set $F \in \mathcal{F}$.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

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Context free or regular ω-languages

(Cohen and Gold 1977; Linna 1976)

Let $L \subseteq \Sigma^\omega$. Then the following propositions are equivalent :

- L is accepted by a Büchi pushdown automaton.
- L is accepted by a Muller pushdown automaton.
- $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$,
  for some context free finitary languages U_i and V_i.
- L is a context free ω-language.

A similar theorem holds if we:

- omit the pushdown stack and replace context free by regular,
- or replace pushdown and context-free by 1-counter.

Possible Extensions

- Timed automata
- Weighted automata
- Probabilistic automata

Languages of infinite words

An ω-language over the alphabet Σ is a subset of Σ^ω.

An ω-language is regular iff it is accepted by a Büchi automaton.

An ω-language is context free iff it is accepted by a Büchi pushdown automaton.

A 1-counter ω-language is an ω-language which is accepted by a 1-counter Büchi automaton.

Complexity of ω-languages

The question naturally arises of the complexity of ω-languages accepted by various kinds of automata.

A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Borel Hierarchy

Σ_1^0 is the class of open subsets of Σ^ω,

Π_1^0 is the class of closed subsets of Σ^ω,

for any integer $n \geq 1$:

Σ_{n+1}^0 is the class of countable unions of Π_n^0-subsets of Σ^ω.

Π_{n+1}^0 is the class of countable intersections of Σ_n^0-subsets of Σ^ω.

Π_{n+1}^0 is also the class of complements of Σ_{n+1}^0-subsets of Σ^ω.

The Borel hierarchy is also defined for levels indexed by countable ordinals.

For any countable ordinal $\alpha \geq 2$:

Σ^0_α is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.

Π^0_α is the class of complements of Σ^0_α-sets

$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$.

Borel Hierarchy

Below an arrow $\longrightarrow$ represents a strict inclusion between Borel classes.

$$
\begin{array}{ccccccccccc}
 & & \Pi^0_1 & & & & & \Pi^0_\alpha & & \Pi^0_{\alpha+1} & \\
 & \nearrow & & \searrow & \nearrow & & \nearrow & & \searrow & \nearrow & \\
\Delta^0_1 & & & & \Delta^0_2 & \cdots\;\cdots & \Delta^0_\alpha & & \Delta^0_{\alpha+1} & & \cdots \\
 & \searrow & & \nearrow & \searrow & & \searrow & & \nearrow & \searrow & \\
 & & \Sigma^0_1 & & & & & \Sigma^0_\alpha & & \Sigma^0_{\alpha+1} &
\end{array}
$$

A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ where ω_1 is the first uncountable ordinal.

Beyond the Borel Hierarchy

There are some subsets of Σ^ω which are not Borel. Beyond the Borel hierarchy is the projective hierarchy.

The class of Borel subsets of Σ^ω is strictly included in the class Σ^1_1 of analytic sets which are obtained by projection of Borel sets.

A set $E \subseteq \Sigma^\omega$ is in the class Σ^1_1 iff :

$$\exists F \subseteq (\Sigma \times \{0,1\})^\omega \text{ such that } F \text{ is } \Pi^0_2 \text{ and}$$

$$E \text{ is the projection of } F \text{ onto } \Sigma^\omega$$

A set $E \subseteq \Sigma^\omega$ is in the class Π^1_1 iff $\Sigma^\omega - E$ is in Σ^1_1.

Suslin's Theorem states that : Borel sets = $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Complete Sets

A set $E \subseteq \Sigma^\omega$ **is $\mathcal{C}$-complete**, where $\mathcal{C}$ is a Borel class Σ^0_α or Π^0_α

or the class Σ^1_1, **for reduction by continuous functions** iff :

$\forall F \subseteq \Gamma^\omega \quad F \in \mathcal{C}$ iff :

$\exists f$ continuous, $f : \Gamma^\omega \to \Sigma^\omega$ such that $F = f^{-1}(E)$

$(x \in F \leftrightarrow f(x) \in E)$.

Example : $\{\sigma \in \{0,1\}^\omega \mid \exists^\infty i \;\; \sigma(i) = 1\}$ is a Π^0_2-complete-set and it is accepted by a deterministic Büchi automaton.

More Examples of Complete Sets

Examples :

$\{\sigma \in \{0,1\}^\omega \mid \exists i \;\; \sigma(i) = 1\}$ is a Σ_1^0**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \forall i \;\; \sigma(i) = 1\} = \{1^\omega\}$ is a Π_1^0**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \exists^{<\infty} i \;\; \sigma(i) = 1\}$ is a Σ_2^0**-complete-set**.

All these ω-languages are ω-regular.

Complexity of ω-languages of deterministic machines

deterministic finite automata (Landweber 1969)

- ω-**regular languages** accepted by **deterministic Büchi automata** are Π^0_2-sets.
- ω-**regular languages** are boolean combinations of Π^0_2-sets hence Δ^0_3-sets.

deterministic Turing machines

- ω-**languages** accepted by **deterministic Büchi Turing machines** are Π^0_2-sets.
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Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi or Muller Turing machines accept **effective analytic sets** (Staiger). The class **Effective-Σ_1^1** of **effective analytic sets** is obtained as the class of **projections of arithmetical sets** and **Effective-Σ_1^1** $\subsetneq \Sigma_1^1$.

Let ω_1^{CK} be the first non recursive ordinal.

Topological Complexity of Effective Analytic Sets

- There are some Σ_1^1-complete sets in **Effective-Σ_1^1**.
- For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ_α^0-complete and some Π_α^0-complete ω-languages in the class **Effective-Σ_1^1**.
- **(Kechris, Marker and Sami 1989)**
The supremum of the set of Borel ranks of **Effective-Σ_1^1-sets** is a countable ordinal $\gamma_2^1 > \omega_1^{\mathrm{CK}}$.

Topological complexity of 1-counter or context free ω-languages

Let $1 - CL_\omega$ be the class of real-time 1-counter ω-languages.

Let $\mathcal{C}$ be a class of ω-languages such that:

$$1 - CL_\omega \subseteq \mathcal{C} \subseteq \textbf{ Effective-}\Sigma^1_1.$$

(a) **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class $\mathcal{C}$.**

(b) **(F. 2005)** The Borel hierarchy of **the class $\mathcal{C}$** is equal to the Borel hierarchy of **the class Effective-Σ^1_1.**

(c) γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class $\mathcal{C}$.**

(d) For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class $\mathcal{C}$.**

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- **(a)** **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class** $\mathcal{C}$.
- **(b)** **(F. 2005)** The Borel hierarchy of **the class** $\mathcal{C}$ is equal to the Borel hierarchy of **the class Effective-Σ^1_1.**
- **(c)** γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class** $\mathcal{C}$.
- **(d)** For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class** $\mathcal{C}$.

Sketch of the proof

It is well known that every Turing machine can be simulated by a (non real time) 2-counter automaton.

Sketch of the proof

First, from a 2-counter automaton A accepting an ω-language $L \subseteq X^\omega$, we construct a real-time 8-counter Büchi automaton B accepting an ω-language of the same topological complexity.

First, we add a storage type called a queue to a 2-counter Büchi automaton in order to read ω-words in real-time.

Then the queue can be simulated by

- two pushdown stacks or
- four counters,
  because each pushdown stack may be simulated by two counters.

Sketch of the proof

This simulation is not done in real-time but one can bound the number of transitions needed to simulate the queue. This allows to pad the strings in L with enough extra letters so that the new language $\theta_S(L)$ will be read in real-time by a 8-counter Büchi automaton.

The padding is obtained via the function $\theta_S : X^\omega \to (X \cup \{E\})^\omega$, where $S = (3k)^3$, with $k = card(X) + 2$:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n{+}1).E^{S^{n+1}}\ldots$$

The ω-language $\theta_S(L)$ is accepted in real time by a Büchi automaton with $2 + 4 + 2 = 8$ counters.

Sketch of the proof

The next step is to simulate a *real-time* 8-counter Büchi automaton $\mathcal{A}$, by a *real-time* 1-counter Büchi automaton $\mathcal{B}$.

The eight first prime numbers are $2; 3; 5; 7; 11; 13; 17; 19$.

We code the content $(c_1, c_2, \ldots, c_8)$ of eight counters by the product $2^{c_1} \times 3^{c_2} \times \ldots \times (17)^{c_7} \times (19)^{c_8}$.

Then we code ω-words in $Y = X \cup \{E\}$ by ω-words in $Z = Y \cup \{A, B, 0\}$.

The new ω-words will have a **special shape** which will allow the propagation of the values of the counters of $\mathcal{A}$.

Sketch of the proof

The product of the eight first prime numbers is:

$$K = 9699690$$

An ω-word $x \in Y^\omega$ is coded by the ω-word

$$h(x) = A.0^K.x(1).B.0^{K^2}.A.0^{K^2}.x(2).B.\ldots B.0^{K^n}.A.0^{K^n}.x(n).B\ldots$$

If $L(\mathcal{A}) \subseteq Y^\omega$ is accepted by a real time 8-counter Büchi automaton $\mathcal{A}$, then one can construct from $\mathcal{A}$ a 1-counter Büchi automaton $\mathcal{B}$, reading words over $Y \cup \{A, B, 0\}$, such that:

$$\forall x \in Y^\omega \quad h(x) \in L(\mathcal{B}) \longleftrightarrow x \in L(\mathcal{A})$$

Sketch of the proof

The mapping $h: Y^\omega \to (Y \cup \{A, B, 0\})^\omega$ is continuous.

The complement $h(Y^\omega)^-$ of the ω-language $h(Y^\omega)$ is an open subset of $(Y \cup \{A, B, 0\})^\omega$ and is accepted by a real time 1-counter automaton.

Thus the ω-language

$$h(L(\mathcal{A})) \cup h(Y^\omega)^- = L(\mathcal{B}) \cup h(Y^\omega)^-$$

is in the class $\mathbf{BCL}(1)_\omega$ and it has the same topological complexity as the ω-language $L(\mathcal{A})$.

Decision Problems

Castro and Cucker proved (1989) that many decision problems about ω-languages of Turing machines are highly undecidable, i.e.

located beyond the arithmetical hierarchy.

From their results and from the previous constructions, we can show that some decision problems about ω-languages of 1-counter automata are also highly undecidable.

Some Decision Problems

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two 1-counter automata over the alphabet Σ. Can we decide whether

- $L(\mathcal{C}_1)$ is empty ?
- $L(\mathcal{C}_1)$ is infinite ?
- $L(\mathcal{C}_1) = \Sigma^\omega$?
- $L(\mathcal{C}_1) = L(\mathcal{C}_2)$?
- $L(\mathcal{C}_1) \subseteq L(\mathcal{C}_2)$?
- $L(\mathcal{C}_1)$ is unambiguous ?
- $L(\mathcal{C}_1)$ is Borel ?
- ...

The Analytical Hierarchy

The Analytical Hierarchy is defined for subsets of $\mathbb{N}^l$ where $l \geq 1$ is an integer. It extends the arithmetical hierarchy to more complicated sets.

Theorem (Kleene 1955)

For each integer $n \geq 1$,

(a) $\Sigma^1_n \cup \Pi^1_n \subsetneq \Sigma^1_{n+1} \cap \Pi^1_{n+1}$.

(b) *A set $R \subseteq \mathbb{N}^l$ is in the class Σ^1_n iff its complement is in the class Π^1_n.*

(c) $\Sigma^1_n - \Pi^1_n \neq \emptyset$ *and* $\Pi^1_n - \Sigma^1_n \neq \emptyset$.

Complete Sets

Definition

Given two sets $A, B \subseteq \mathbb{N}$ we say A is 1-reducible to B and write $A \leq_1 B$ if there exists a total computable injective function f from $\mathbb{N}$ to $\mathbb{N}$ such that $A = f^{-1}[B]$.

Definition

A set $A \subseteq \mathbb{N}$ is said to be Σ^1_n-complete (respectively, Π^1_n-complete) iff A is a Σ^1_n-set (respectively, Π^1_n-set) and for each Σ^1_n-set (respectively, Π^1_n-set) $B \subseteq \mathbb{N}$ it holds that $B \leq_1 A$.

Some differences between Turing machines and 1-counter automata

Theorem (Castro and Cucker 1989)

The non-emptiness problem and the infiniteness problem for ω-languages of Turing machines are Σ^1_1-complete.

Theorem (Cohen and Gold 1977)

The non-emptiness problem and the infiniteness problem for ω-languages of 1-counter Büchi automata are decidable.

Proof. An ω-language L is accepted by a 1-counter Büchi automaton iff it is of the form $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$, for some 1-counter finitary languages U_i and V_i. The emptiness problem for 1-counter (and even context-free) finitary languages is decidable.

Some similarities

Theorem (Castro and Cucker 1989; F. 2009)

The following problems are Π^1_2-complete for ω-languages of Turing machines and for ω-languages of 1-counter Büchi automata:

1. *The universality problem.*
2. *The inclusion problem.*
3. *The equivalence problem.*
4. *The cofiniteness problem.*

Ambiguity of automata and languages

A Büchi Turing machine $\mathcal{T}$ over the alphabet Σ is unambiguous if every ω-word $x \in \Sigma^\omega$ has at most one accepting run by $\mathcal{T}$

A 1-counter Büchi automaton $\mathcal{C}$ over the alphabet Σ is unambiguous if every ω-word $x \in \Sigma^\omega$ has at most one accepting run by $\mathcal{C}$.

An ω-language of a Turing machine is unambiguous iff it is accepted by an unambiguous Turing machine. Otherwise it is inherently ambiguous.

An ω-language of a 1-counter Büchi automaton is unambiguous iff it is accepted by an unambiguous 1-counter Büchi automaton.

The Unambiguity problem

We denote by $\mathcal{C}_z$ the 1-counter Büchi automaton of index z.

Theorem (F. 2009)

The unambiguity problem for ω-languages of 1-counter Büchi automata is Π^1_2-complete, i.e. :

$$\{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is unambiguous}\} \text{ is } \Pi^1_2\text{-complete.}$$

Proof. We first express by a Π^1_2-formula that an ω-language of a 1-counter Büchi automaton is unambiguous.

We reduce the universality problem for ω-languages of Turing machines to this problem, using topological properties, to prove the completeness part of the result.

Some undecidable problems higher in the analytical hierarchy

Some decisions problems for ω-languages of Turing machines and for ω-languages of 1-counter Büchi automata are located

above the two first levels of the analytical hierarchy.

We can use Set Theory to obtain such lower bounds of decision problems.

Perfect Sets, Thin Sets

Definition

Let $P \subseteq \Sigma^\omega$, where Σ is a finite alphabet having at least two letters. The set P is a perfect subset of Σ^ω iff it is a non-empty closed set which has no isolated points.

A perfect subset of Σ^ω has cardinality $2^{\aleph_0}$.

Definition

A set $X \subseteq \Sigma^\omega$ is said to be thin iff it contains no perfect subset.

Theorem (Souslin)

*(**ZFC**) An analytic set $X \subseteq \Sigma^\omega$ is either countable or contains a perfect subset. Thus every thin analytic set is countable.*

This result is not true for co-analytic sets in **ZFC**. We need additional axioms like analytic determinacy.

The constructible sets

The class $\mathbf{L}$ of *constructible sets* in a model $\mathbf{V}$ of **ZF** is defined by

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}(\alpha)$$

where the sets $\mathbf{L}(\alpha)$ are constructed by induction as follows:

1. $\mathbf{L}(0) = \emptyset$
2. $\mathbf{L}(\alpha) = \bigcup_{\beta < \alpha} \mathbf{L}(\beta)$, for α a limit ordinal, and
3. $\mathbf{L}(\alpha + 1)$ is the set of subsets of $\mathbf{L}(\alpha)$ which are definable from a finite number of elements of $\mathbf{L}(\alpha)$ by a first-order formula relativized to $\mathbf{L}(\alpha)$.

If **V** is a model of **ZF** and **L** is the class of *constructible sets* of **V**, then the class **L** forms a model of **ZFC + CH.** Notice that the axiom (**V=L**) means "every set is constructible" and that it is consistent with **ZFC.**

Theorem (Kechris 1975; Guaspari, Sacks)

*(**ZFC**) Let Σ be a finite alphabet having at least two letters. There exists a thin Π_1^1-set $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ which contains every thin, Π_1^1-subset of Σ^ω. It is called the largest thin Π_1^1-set in Σ^ω.*

Theorem (Kechris 1975; Guaspari, Sacks)

*(**ZFC**) The cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal of $\omega_1^{\mathbf{L}}$.*

This means that in a given model **V** of **ZFC** the cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal *in* **V** of the ordinal $\omega_1^{\mathbf{L}}$ which is the first uncountable ordinal in the inner model **L** of constructible sets of **V**.

$\omega_1^{\mathbf{L}} \leq \omega_1$: either $\omega_1^{\mathbf{L}} = \omega_1$ or $\omega_1^{\mathbf{L}}$ is countable.

The Largest Thin Effective Coanalytic Set

Theorem

1. *(**ZFC + V=L**) The largest thin Π^1_1-set in Σ^ω is not a Borel set.*
2. *(**ZFC** $+ \omega_1^{\mathbf{L}} < \omega_1$) The largest thin Π^1_1-set in Σ^ω is countable, hence a Σ^0_2-set.*

Proof. In (**ZFC + V=L**) it holds that $\omega_1 = \omega_1^{\mathbf{L}}$. Thus the set $\mathcal{C}_1(\Sigma^\omega)$ has cardinal ω_1 and it is not countable. But it is thin, hence has no perfect subset. Thus it cannot be a Borel set because Borel sets have the perfect set property.

(**ZFC** $+ \omega_1^{\mathbf{L}} < \omega_1$) the ordinal $\omega_1^{\mathbf{L}}$ is countable so the set $\mathcal{C}_1(\Sigma^\omega)$ is countable. It is a countable union of singletons, and each singleton is a closed set. Thus $\mathcal{C}_1(\Sigma^\omega)$ is a countable union of closed sets, i.e. a Σ^0_2-set.

From effective coanalytic sets to 1-counter automata

The complement of $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ is an effective analytic set accepted by a Büchi Turing machine $\mathcal{T}$.

We can now use previous constructions to obtain:

- A 2-counter Büchi automaton $\mathcal{A}_1$,
- A real time 8-counter Büchi automaton $\mathcal{A}_2$,
- A real time 1-counter Büchi automaton $\mathcal{A}_3$,

such that $L(\mathcal{T})$, $L(\mathcal{A}_1)$, $L(\mathcal{A}_2)$, and $L(\mathcal{A}_3)$, all have the same toplogical complexity.

The Topological complexity of a 1-counter ω-language depends on the models of ZFC

Theorem (F. 2009)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC**.

1. (**ZFC + V=L**). *The ω-language $L(\mathcal{A})$ is a true analytic set.*
2. (**ZFC** $+ \omega_1^{\mathbf{L}} < \omega_1$). *The ω-language $L(\mathcal{A})$ is a Π_2^0-set.*

More Independence results

Theorem

There exists a real-time 1-counter Büchi automaton $\mathcal{A}$ such that the cardinality of the complement $L(\mathcal{A})^-$ of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC***:*

1. *There is a model V_1 of* **ZFC** *in which $L(\mathcal{A})^-$ is countable.*
2. *There is a model V_2 of* **ZFC** *in which $L(\mathcal{A})^-$ has cardinal $2^{\aleph_0}$.*
3. *There is a model V_3 of* **ZFC** *in which $L(\mathcal{A})^-$ has cardinal $\aleph_1$ with $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.*

Using Shoenfield's Absoluteness Theorem and the preceding proof we can prove the following result:

Theorem (F. 2009)

Let α be a countable ordinal. Then

1. *For $\alpha > 2$,* $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is in the Borel class* $\Sigma^0_\alpha\}$ *is not in* $(\Pi^1_2 \cup \Sigma^1_2)$.
2. *For $\alpha \geq 2$,* $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is in the Borel class* $\Pi^0_\alpha\}$ *is not in* $(\Pi^1_2 \cup \Sigma^1_2)$.
3. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is a Borel set* $\}$ *is not in* $(\Pi^1_2 \cup \Sigma^1_2)$.

Complexity of decision problems

We first prove item (1). Let $\mathcal{A}$ be the real-time 1-counter Büchi automaton cited in preceding theorem and let z_0 be its index so that $\mathcal{A} = \mathcal{C}_{z_0}$.

Assume now that **V** is a model of (**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$). In the model **V**, the ω-language $L(\mathcal{A})$ is a Π_2^0-set, hence also a Σ_α^0-set for any countable ordinal $\alpha > 2$. Thus, for $\alpha > 2$, the integer z_0 belongs to the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\Sigma_\alpha^0\}$.

But in the inner model $\mathbf{L} \subseteq \mathbf{V}$, the ω-language $L(\mathcal{A})$ is an analytic but non Borel set so the integer z_0 does not belong to the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\Sigma_\alpha^0\}$.

On the other hand, Shoenfield's Absoluteness Theorem implies that every Σ^1_2-set (respectively, Π^1_2-set) is absolute for all inner models of **ZFC**.

In particular, if the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\Sigma^0_\alpha\}$ was a Π^1_2-set or a Σ^1_2-set, then it could not be a different subset of $\mathbb{N}$ in the models **V** and **L** considered above. Therefore, for any countable ordinal $\alpha > 2$, the set

$$\{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is in the Borel class } \Sigma^0_\alpha\}$$

is not in $(\Pi^1_2 \cup \Sigma^1_2)$.

Infinitary rational relations

(Gire 1981, Gire and Nivat 1984)

A set $R \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation iff one the two following equivalent conditions holds :

- R is recognized by a Büchi transducer $\mathcal{T}$:

  R is the set of pairs $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u is the input word and v is the output word of a successful computation of $\mathcal{T}$.
- R is accepted by a 2-tape Büchi automaton $\mathcal{A}$ with two asynchronous reading heads.

Infinitary rational relations

A set $R \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation iff it is generated from :

- the empty set $\emptyset$, and
- singletons $\{(a, \lambda)\}, \{(\lambda, b)\}$, $a \in \Sigma, b \in \Gamma$, where λ is the empty word.

by operations of

- finite union,
- concatenation product : $(u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2)$
- star operation,
- operation $R \rightarrow R^\omega$ over finitary rational relations.

Notice that an infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ may be seen as an ω-language $R \subseteq (\Sigma \times \Gamma)^\omega$ over the alphabet $\Sigma \times \Gamma$.

Similar results for 2-tape Büchi automata

Infinitary rational relations have same topological complexity as ω-languages accepted by real-time 1-counter Büchi automata or by Büchi Turing machines (i.e. effective analytic sets). And:

Theorem (F. 2009)

The topological complexity of an ω-language accepted by a 2-tape Büchi automaton is not determined by the axiomatic system **ZFC**. *Indeed there is a 2-tape Büchi automaton $\mathcal{B}$ such that:*

1. *There is a model V_1 of* **ZFC** *in which the ω-language $L(\mathcal{B})$ is an analytic but non Borel set.*
2. *There is a model V_2 of* **ZFC** *in which the ω-language $L(\mathcal{B})$ is a Π^0_2-set.*

We consider now transition systems wich are extensions of finite automata and can have countably many states.

Transition systems

A Büchi transition system is a tuple $\mathcal{T} = (\Sigma, Q, \delta, q_0, Q_f)$, where Σ is a finite input alphabet, Q is a countable set of states, $q_0 \in Q$ is the initial state, $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $Q_f \subseteq Q$ is the set of final states.

A run of $\mathcal{T}$ over an infinite word $\sigma \in \Sigma^\omega$ is an infinite sequence of states $(t_i)_{i \geq 0}$, such that :
$t_0 = q_0$, and for each $i \geq 0$, $(t_i, \sigma(i+1), t_{i+1}) \in \delta$.

The run is said to be accepting iff there are infinitely many integers i such that t_i is in Q_f.

The transition system is said to be finitely branching if for each state $q \in Q$ and each $a \in \Sigma$, there are only finitely many states q' such that $(q, a, q') \in \delta$.

The transition system is unambiguous if each infinite word $\sigma \in \Sigma^\omega$ has at most one accepting run by $\mathcal{T}$.

Using Lusin and Souslin's Theorem, Arnold proved that:

Theorem (Arnold 1983)

1. *The analytic subsets of Σ^ω are the subsets of Σ^ω which are accepted by finitely branching Büchi transition systems.*
2. *The Borel subsets of Σ^ω are the subsets of Σ^ω which are accepted by unambiguous finitely branching Büchi transition systems.*

Is there an effective analogue to Arnold's Theorem?

We know that effective analytic subsets of Σ^ω are those which are accepted by Turing machines (Staiger).

Question

Are the sets which are effective analytic and Borel those which are accepted by unambiguous Büchi Turing machines ?

The answer is no:

Theorem (F. 2010)

It is consistent with **ZFC** *that there is an effective analytic subset of Σ^ω in the Borel class Π^2_0 which is not accepted by any unambiguous Turing machine.*

Sketch of the proof

Recall the following result:

Theorem (F. 2009)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC**.

1. *(**ZFC + V=L**). The ω-language $L(\mathcal{A})$ is a true analytic set.*
2. *(**ZFC** $+ \omega_1^{\mathbf{L}} < \omega_1$). The ω-language $L(\mathcal{A})$ is a Π_2^0-set.*

Sketch of the proof

Using the fact that the image of a Borel set by an injective continuous function is a Borel set we can show that **every unambiguous ω-language is a Borel set.**

Thus in (**ZFC + V=L**) the ω-language $L(\mathcal{A})$ is a true analytic set, hence it is inherently ambiguous.

But in (**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$), the ω-language $L(\mathcal{A})$ is a Borel set in the class Π_2^0.

Sketch of the proof

Let now **V** be a model of (**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$). In this model $L(\mathcal{A})$ is a Borel set in the class Π_2^0.

Towards a contradiction, assume that this ω-language is accepted by an unambiguous Büchi Turing machine $\mathcal{T}$. The property $L(\mathcal{A}) = L(\mathcal{T})$ is a Π_2^1-property and the property that the Büchi Turing machine $\mathcal{T}$ is unambiguous is also a Π_2^1-property.

Thus by Shoenfield's Absoluteness Theorem it would hold in the inner model of constructible sets $\mathbf{L} \subset \mathbf{V}$ that $L(\mathcal{A})$ is also accepted by an unambiguous Büchi Turing machine $\mathcal{T}$.

But the inner model $\mathbf{L} \subset \mathbf{V}$ is also a model of (**ZFC + V=L**). Thus in **L** the ω-language $L(\mathcal{A})$ is an analytic but non Borel set. Thus it cannot be accepted by any unambiguous Büchi Turing machine.

Sketch of the proof

Thus in the model V the ω-language $L(\mathcal{A})$ is a Borel set in the class Π^0_2 which can not be accepted by any unambiguous Büchi Turing machine.

Notice that the axiom of analytic determinacy or the axiom of the existence of a measurable cardinal imply that $\omega_1^{\mathbf{L}} < \omega_1$, hence the existence of this ω-language $L(\mathcal{A})$ which is Π^0_2 and inherently ambiguous.

An effective analogue to Arnold's Theorem

Using some effective descriptive set theory one proves:

Theorem (F. 2011)

The Effective-Δ_1^1 subsets of Σ^ω are those which are accepted by unambiguous Büchi Turing machines.

In particular the following corollary holds:

Theorem (F. 2011)

If L is an effective analytic set which is a Borel set of rank $\alpha > \omega_1^{\mathrm{CK}}$, then L is inherently ambiguous.

A dichotomy result for ω-languages of Turing machines

Theorem (F. 2011)

Let L be an ω-language accepted by a Büchi Turing machine $\mathcal{T}$. Then Either
(1) The ω-language $L = L(\mathcal{T})$ is an effective Δ_1^1-set, and hence it is unambiguous, or
(2) for every Büchi Turing machine $\mathcal{T}'$ accepting L there are infinitely many ω-words having $2^{\aleph_0}$ accepting runs by $\mathcal{T}'$, so L has "a great degree of ambiguity".

Non-Borel ω-languages of Turing machines

Theorem (F. 2011, F. and Simonnet 2005 for a similar result for pushdown automata)

Let L be an ω-language accepted by a Büchi Turing machine $\mathcal{T}$ which is an analytic but non-Borel set. Then for every Büchi Turing machine $\mathcal{T}'$ accepting L there are $2^{\aleph_0}$ ω-words having $2^{\aleph_0}$ accepting runs by $\mathcal{T}'$, so L has "the maximum degree of ambiguity".

Theorem (F. 2011)

It is consistent with **ZFC** *that there exists an ω-language accepted by a Büchi Turing machine $\mathcal{T}$ in the Borel class Π^2_0 which has "the maximum degree of ambiguity".*

Open Question

Does there exist such a Borel set having the maximum degree of ambiguity in every model of ZFC ?

Conjecture: Yes

Open Questions

Theorem [F. (2005)]
There are ω-languages accepted by Büchi 1-counter automata of every Borel rank of an effective analytic set.

Theorem [Kechris, Marker, and Sami (1989)]
The supremum of the set of Borel ranks of effective analytic sets is the ordinal $\gamma_2^1 > \omega_1^{\mathrm{CK}}$.

Every ω-language accepted by a Büchi 1-counter automaton can be written as a finite union $L = \bigcup_{1 \leq i \leq n} U_i.V_i^{\omega}$, **where for each integer i, U_i and V_i are 1-counter languages.**

Conjecture **From these results it seems plausible that there exist some ω-powers of languages accepted by 1-counter automata which have Borel ranks up to the ordinal γ_2^1, although these languages are located at the very low level in the complexity hierarchy of finitary languages.**

Open question

There is a 1-counter ω-language $L(\mathcal{A})$ which is Borel in some model of **ZFC** and non Borel in some other model of **ZFC**.

But

$$L(\mathcal{A}) = \bigcup_{1 \leq i \leq n} U_i . V_i^{\omega}$$

for some finitary 1-counter-languages U_i and V_i.

When $L(\mathcal{A})$ is non Borel then at least one ω-power language V_i^{ω} is non Borel.

Are all V_i^{ω} Borel in the other case ?

Does the topological complexity of the ω-power of a finitary 1-counter-language depend on the model of **ZFC**?

The ordinal γ_2^1 may depend on set theoretic axioms

The ordinal γ_2^1 is the least basis for subsets of ω_1 which are Π_2^1 in the codes.

It is the least ordinal such that whenever $X \subseteq \omega_1$, $X \neq \emptyset$, and $\hat{X} \subseteq WO$ is Π_2^1, there is $\beta \in X$ such that $\beta < \gamma_2^1$.

The least ordinal which is not a Δ_n^1-ordinal is denoted δ_n^1.

Theorem (Kechris, Marker and Sami 1989)

- (ZFC) $\delta_2^1 < \gamma_2^1$
- (V = L) $\gamma_2^1 = \delta_3^1$
- (Π_1^1-Determinacy) $\gamma_2^1 < \delta_3^1$

Are there effective analytic sets of every Borel rank $\alpha < \gamma_2^1$?

The Analytical Hierarchy

Let $k, l > 0$ be some integers and $R \subseteq \mathcal{F}^k \times \mathbb{N}^l$, where $\mathcal{F}$ is the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$.
The relation R is said to be recursive if its characteristic function is recursive.

A subset R of $\mathbb{N}^l$ is analytical if it is recursive or if there exists a recursive set $S \subseteq \mathcal{F}^m \times \mathbb{N}^n$, with $m \geq 0$ and $n \geq l$, such that $(x_1, \ldots, x_l)$ is in R iff

$$(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$$

where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq m + n - l$, and where $s_1, \ldots, s_{m+n-l}$ are $f_1, \ldots, f_m, x_{l+1}, \ldots, x_n$ in some order.

$(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$ is called a predicate form for R.
The reduced prefix is the sequence of quantifiers obtained by suppressing the quantifiers of type 0 from the prefix.

The Analytical Hierarchy

For $n > 0$, a Σ_n^1-prefix is one whose reduced prefix begins with $\exists^1$ and has $n-1$ alternations of quantifiers. For $n > 0$, a Π_n^1-prefix is one whose reduced prefix begins with $\forall^1$ and has $n-1$ alternations of quantifiers.

A Π_0^1-prefix or Σ_0^1-prefix is one whose reduced prefix is empty.

A predicate form is a Σ_n^1 (Π_n^1)-form if it has a Σ_n^1 (Π_n^1)-prefix.

The class of sets in $\mathbb{N}^l$ which can be expressed in Σ_n^1-form (respectively, Π_n^1-form) is denoted by Σ_n^1 (respectively, Π_n^1).

The class $\Sigma_0^1 = \Pi_0^1$ is the class of arithmetical sets.